Susceptible, Infected, Recovered (SIR) Model for Epidemics

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Disclaimer; does not constitute engineering advice or detailed predictive capability. It is for educational and illustrative applications only, to demonstrate and understand the effects of countermeasures such as social distancing, vaccinations, barriers, face masks, and so on against diseases such as COVID-19 and seasonal flu. This material accompanies an Excel spreadsheet in which the user can try different values for a disease's basic reproduction number (R_0), recovery rate, and vaccinated fraction of a population (if a vaccination is available) to:

- Illustrate the curve everybody is trying to flatten
- Show how diligent compliance with recommended countermeasures such as social distancing can "break the curve," i.e. suppress its first derivative to zero right out of the starting gate, to teach the importance of compliance with the countermeasures in question
- Show how widespread vaccination (available for the seasonal flu) can have the same effect, thus encouraging people to get the annual vaccine

Susceptible, Infected, Recovered (SIR) Model

The Susceptible, Infected, Recovered (SIR) model[1] predicts the course of an epidemic as follows. Everybody in a population of N falls into one of the following categories:

- Susceptible (S): people who have no resistance to the disease
- Infected (I): people who have the disease and are contagious to others
- Recovered (R): people who have recovered from the disease and are no longer susceptible or contagious, or who have been vaccinated against it and are not able to become contagious.
- S+I+R = N at all times. Alternatively, the fractions S/N + I/N + R/N must add to 1 or 100%.

The rates of change for these populations are as follows:

Depletion of Susceptible population due to infection	$$\frac{dS}{dt} = -\beta I \frac{S}{N}$$ where β (beta) is the *transmission rate* in people per infected person per unit time.
Increase in Recovered population	$$\frac{dR}{dt} = -\gamma I$$ where γ (gamma) is the *recovery rate* in infected people per day. It is the reciprocal of the period during which an infected person remains contagious regardless of whether he or she exhibits symptoms.
Rate of change in the Infected population (equals the rate of depletion of the Susceptible	$$\frac{dI}{dt} = \beta I \frac{S}{N} - \gamma I$$

population minus the rate of increase in the Recovered population)

Also, $\beta = R_0 \gamma$ where R_0 is the basic reproduction number, or average number of susceptible people to whom an infected person will transmit the disease while he or she is contagious.

R_0 for common diseases[2]

- Ebola: 1.51 to 2.53
- Seasonal flu: 2 to 3
- Measles: 12 to 18
- Polio: 5 to 7 (hence the terrifying nature of this disease prior to the Salk vaccine)
- Smallpox: 5 to 7. This disease was similarly terrifying prior to Edward Jenner's vaccine.
- COVID-19: 1.4 to 4.08 (2.6 seems to be the current best guess.)

As an example, suppose $R_0 = 3$ which means that, on average, an infected person will infect three susceptible people while he or she is contagious for 15 days. (The contagious period for seasonal flu is considerably less, and that for COVID-19 seems to be around 14 days including the asymptomatic period.) Then:

$$\gamma = \frac{1}{15\ \ days}$$

$$\beta = 3\frac{people}{infected\ person}\frac{1}{15\ \ days} = 0.2\frac{people}{infected\ person\quad day}$$

That is, the transmission rate is 0.2 people *per infected person per day*. We can then divide each of the differential equations by N (the total population) to express the SIR model into something very similar to a chemical kinetics problem in which the reaction rates are proportional to *rate constants* times *concentrations* of reactants, except in this case the concentrations are given in fractions of the total population rather than gram-moles per liter or pound-moles per gallon.

$$\frac{d\frac{S}{N}}{dt} = -\beta\frac{I}{N}\frac{S}{N} \quad or \quad \frac{ds}{dt} = -\beta is$$

where $i = I/N$ is the infected fraction of the population, $s = S/N$ is the susceptible fraction, and $r = R/N$ is the recovered and/or vaccinated fraction

$$\frac{d\frac{R}{N}}{dt} = -\gamma\frac{I}{N} \quad or \quad \frac{dr}{dt} = -\gamma i$$

$$\frac{d\frac{I}{N}}{dt} = \beta\frac{I}{N}\frac{S}{N} - \gamma\frac{I}{N} \quad or \quad \frac{di}{dt} = \beta is - \gamma i$$

Also, $\beta = R_0 \gamma$ which means

$$\frac{di}{dt} = \gamma i(R_0 s - 1)$$

which is zero (i is maximized) when

$$\boldsymbol{s = \frac{1}{R_0}}$$

which is the top of the "curve" we are trying to flatten. If for example R_0=3 then the top of the curve will be reached when only 1/3 of the population is still susceptible, whether due to vaccination or having actually had the disease. We can use this relationship to check whether the numerical method shown below is working correctly.

In addition, when R_0<1 (due, for example, to countermeasures like social distancing and deployment of improvised face masks which offer partial protection for activities like grocery shopping), di/dt is negative even when s=1 which means there is never a curve to flatten.

Numerical Integration

The differential equations shown above cannot be solved directly because they are functions of one another; that is, di/dt is a function of not only i but also s, and ds/dt is a function of i. They can however be handled by the Runge-Kutta method.[3] When a reaction rate is given by

$$r(x) = \frac{dx}{dt}, \;\; then\; x_{j+1} = x_j + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$$

where, for a time increment Δt,

$$k_0 = \Delta t \times r(x_j)$$

$$k_1 = \Delta t \times r\left(x_j + \frac{k_0}{2}\right)$$

$$k_2 = \Delta t \times r\left(x_j + \frac{k_1}{2}\right)$$

$$k_3 = \Delta t \times r(x_j + k_2)$$

In this case, if we let the current populations be s, i, and r (no subscript j),

Susceptible

$$s_0 = -\Delta t \beta s i$$

$$s_1 = -\Delta t \beta \left(s + \frac{s_0}{2}\right)\left(i + \frac{i_0}{2}\right)$$

$$s_2 = -\Delta t \beta \left(s + \frac{s_1}{2}\right)\left(i + \frac{i_1}{2}\right)$$

$$s_3 = -\Delta t \beta (s + s_2)(i + i_2)$$

Infected

$$i_0 = +\Delta t \times (\beta s i - \gamma i) = -s_0 - \Delta t \gamma i$$

$$i_1 = \Delta t\left(\beta\left(s + \frac{s_0}{2}\right)\left(i + \frac{i_0}{2}\right) - \gamma\left(i + \frac{i_0}{2}\right)\right)$$

$$= -s_1 - \gamma\left(i + \frac{i_0}{2}\right)$$

$$i_2 = \Delta t\left(\beta\left(s+\frac{s_1}{2}\right)\left(i+\frac{i_1}{2}\right)-\gamma\left(i+\frac{i_1}{2}\right)\right)$$
$$= -s_2 - \gamma\left(i+\frac{i_1}{2}\right)$$
$$i_3 = \Delta t\big(\beta(s+s_2)(i+i_2)-\gamma(i+i_2)\big)$$
$$= -s_3 - \gamma(i+i_2)$$

Recovered

$$r_0 = \Delta t \gamma i$$

$$r_1 = \Delta t \gamma\left(i+\frac{i_0}{2}\right)$$

$$r_2 = \Delta t \gamma\left(i+\frac{i_1}{2}\right)$$

$$r_3 = \Delta t \gamma (i+i_2)$$

And then

$$s_{next} = s + \frac{1}{6}(s_0 + 2s_1 + 2s_2 + s_3)$$

$$i_{next} = i + \frac{1}{6}(i_0 + 2i_1 + 2i_2 + i_3)$$

$$r_{next} = r + \frac{1}{6}(r_0 + 2r_1 + 2r_2 + r_3)$$

Recall that

(1) s+i+r = 1 (100%)
(2) i is maximized when $s=1/R_0$

In addition, at any given time t,

$$r(t) = \int_0^t \frac{di}{dt}$$

which means the *attack rate*, or total number of people who become ill, is the total area under the Infected curve:

$$Attack\ rate = \int_0^\infty \frac{di}{dt}$$

To use the spreadsheet

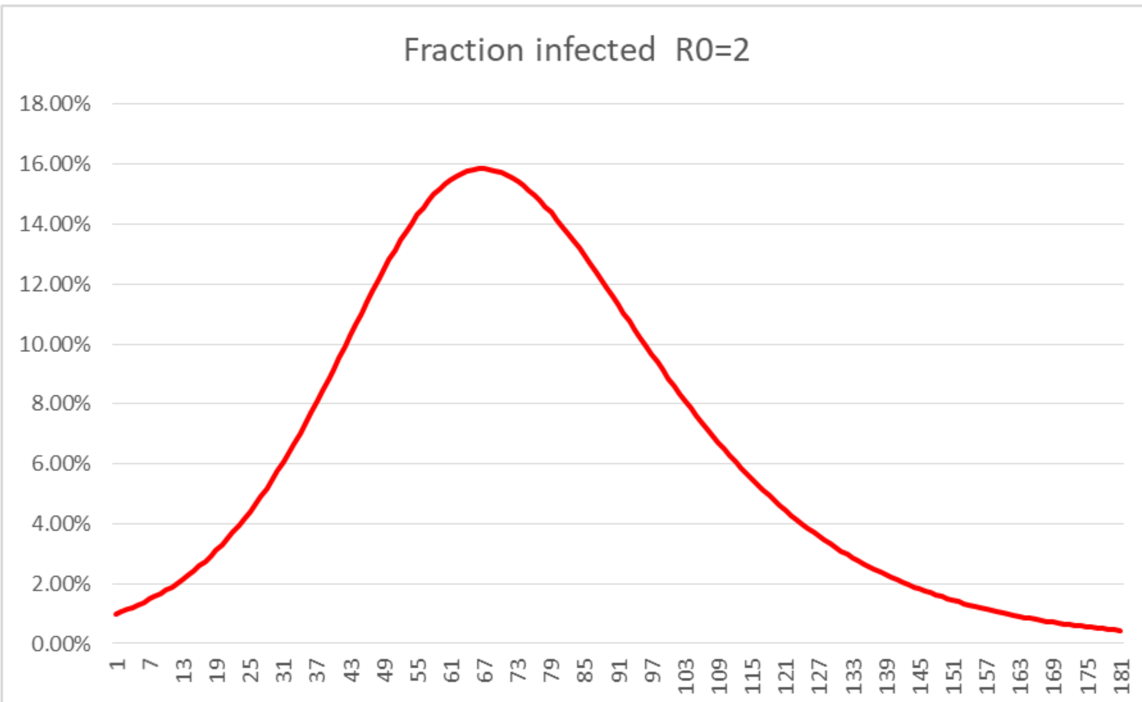
Enter the initial infected fraction (this will not matter too much) and the initial vaccinated fraction, if any. As an example, we can model the seasonal flu with a given basic reproduction number R_0 (also user-defined, along with the recovery period) if, for example, 50% of the people get a 100% effective vaccine or conceivably 50% get a 50% effective vaccine (in which case, use 25% for the fraction of immune people although, in practice, all are likely to have varying degrees of immunity). The resulting figures can be used to advocate for flu vaccination and to (for example) encourage employers and insurers to cover the cost.

The increment in days is the Δt in the Runge-Kutta equations. A one-day increment seems adequate to get good results.

The spreadsheet will calculate the fractions s, i, and r for each time increment and also report the *attack rate* (total percent infected) which is the last result for r and also the area under the curve for the infected fraction, minus the initiated vaccinated fraction if any. Note however that the user must ensure that r has ceased to change significantly in the last row, or else the integration has not been performed completely. (Enough rows have been provided for foreseeable practical purposes although more may be needed for extremely low R_0 values.)

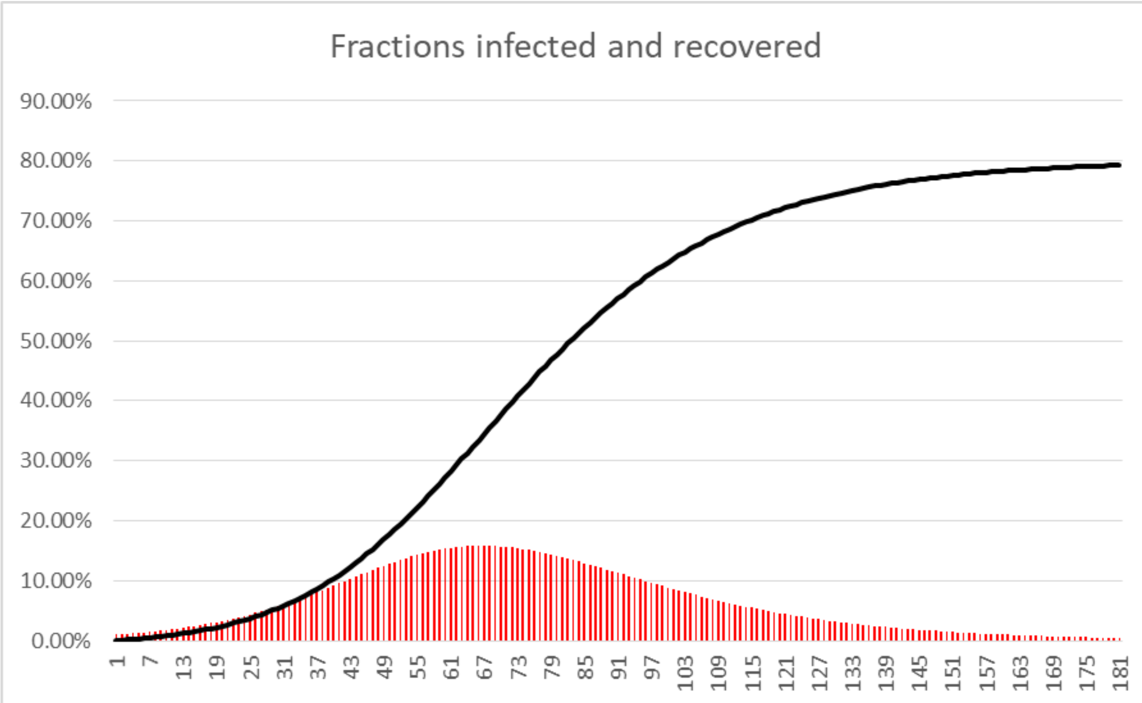
Example: R_0=2, No Vaccination

Starting conditions, **user-defined in yellow**				
Initial susceptible fraction (calculated)	0.99	These three must add to exactly 1		
Initial infected fraction	0.01			
Initial vaccinated fraction	0	Use 0 if no vaccination is available		
R0	2	Basic reproduction number	dI/dt = 0 when S/N =	0.500
recovery period (days)	15			
recovery rate gamma (calculated)	0.067	reciprocal days		
transmission rate beta (calculated)	0.133	infections per day	Attack rate	80.02%
Increment in days (delta t)	1			



The maximum (15.84% infected, 2 rows of the spreadsheet) is reached when the susceptible fractions are 50.73% and 49.66% whose average is 50.2% which compares favorably with the exact value of $1/R_0$ = 50.00%.

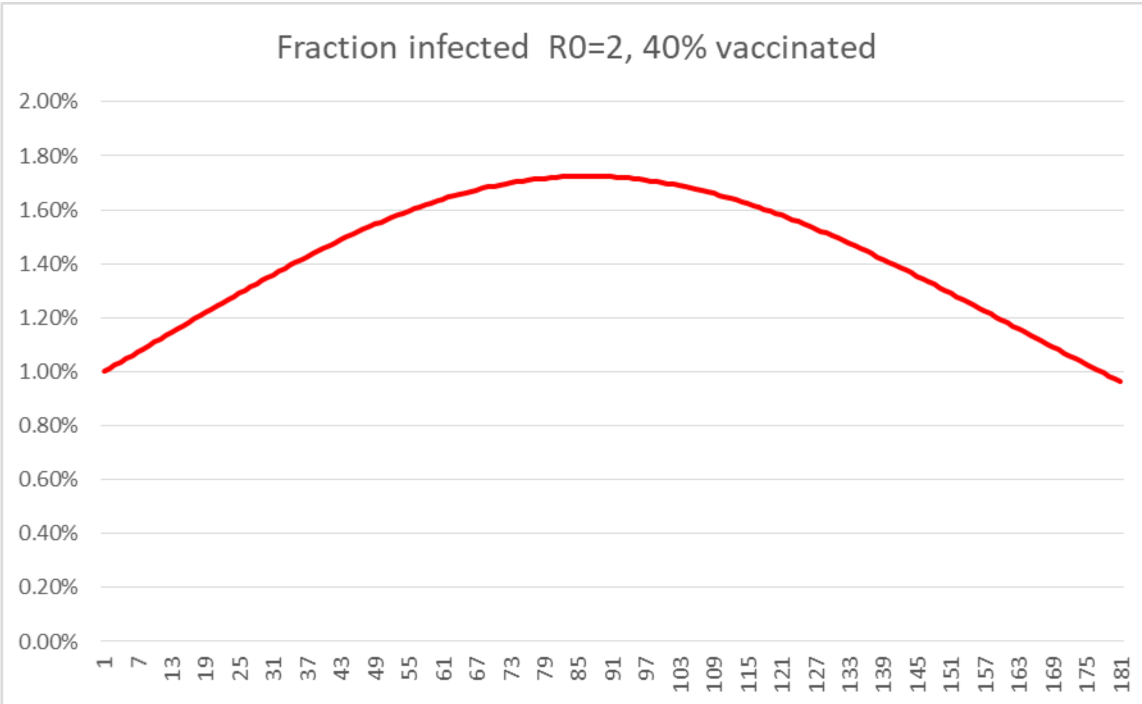
In addition, the *attack rate* (or area under the infected curve is close to 80%. This shows how dangerous a disease with R_0=2 can be if allowed to run out of control.



Example: R_0=2, 40% vaccinated

Starting conditions, **user-defined in yellow**							
Initial susceptible fraction (calculated)	0.59	These three must add to exactly 1					
Initial infected fraction	0.01						
Initial vaccinated fraction	0.40	Use 0 if no vaccination is available					
R0	2	Basic reproduction number		dI/dt = 0 when S/N =			0.500
recovery period (days)	15						
recovery rate gamma (calculated)	0.067	reciprocal days					
transmission rate beta (calculated)	0.133	infections per day		Attack rate		21.59%	
Increment in days (delta t)	1						

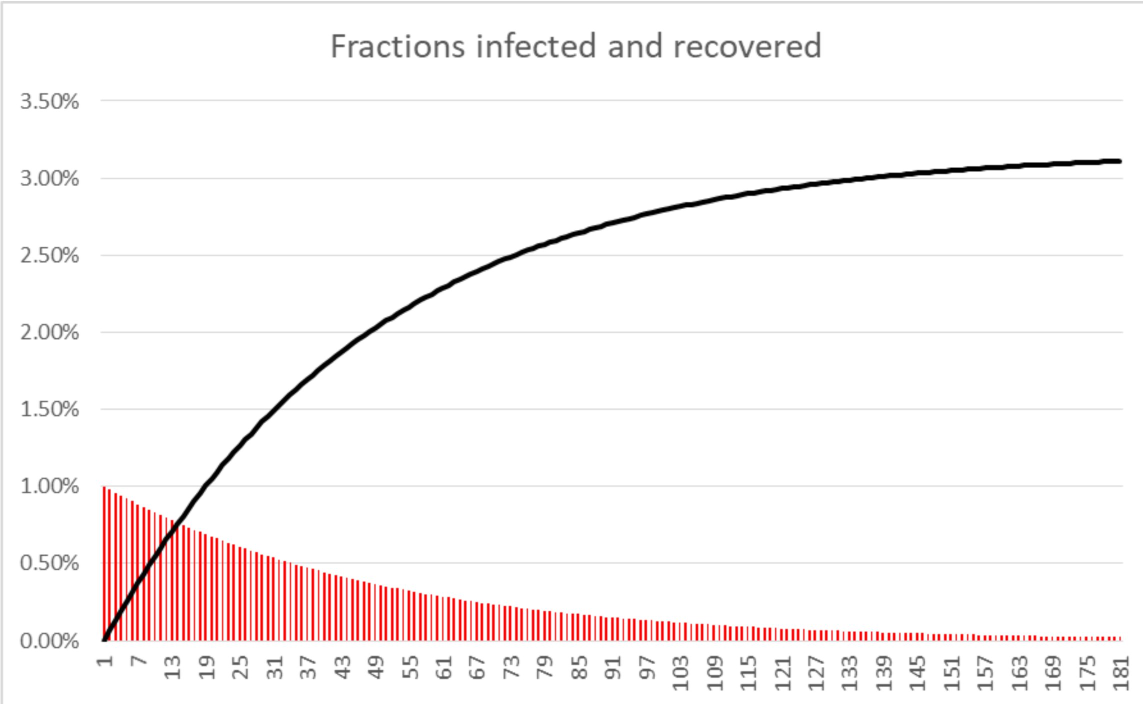
The maximum is reached at only 1.72% and only about 22% of the people (area under the infected curve) rather than 80% get the disease.



Example: No vaccination is available but countermeasures (such as social distancing) reduce R0 to 0.7.

This one can be used to underscore the importance of following the CDC's and Surgeon General's instructions with regard to social distancing. If all the countermeasures reduce R_0 to 0.7 (for example) about 3.2% of the population will eventually get the illness. Note also that S/N (the fraction of the susceptible population at the maximum) is 1.43 which is of course impossible so there is no maximum, as shown in the graph.

Starting conditions, **user-defined in yellow**								
Initial susceptible fraction (calculated)	0.99	These three must add to exactly 1						
Initial infected fraction	0.01							
Initial vaccinated fraction	0.00	Use 0 if no vaccination is available						
R0	0.7	Basic reproduction number			dI/dt = 0 when S/N =			1.429
recovery period (days)	15							
recovery rate gamma (calculated)	0.067	reciprocal days						
transmission rate beta (calculated)	0.047	infections per day			Attack rate		3.18%	
Increment in days (delta t)	1				Area under the Infected curve = total r minus initia			



[1] Ridenhour, Kowalik, and Shay. 2014. "Unraveling R0: Considerations for Public Health Applications." Am J Public Health. 2014 February; 104(2): e32–e41. https://www.ncbi.nlm.nih.gov/pmc/articles/PMC3935673/ explains the SIR number and basic reproduction number R0.

[2] Eisenberg, Joseph. 2020. "R0: How Scientists Quantify the Intensity of an Outbreak Like Coronavirus and Its Pandemic Potential." U. Michigan School of Public Health. https://sph.umich.edu/pursuit/2020posts/how-scientists-quantify-outbreaks.html

[3] Smith, J.M. 1981. *Chemical Engineering Kinetics*. New York; McGraw-Hill